A SHORT TUTORIAL ON LIE ALGEBRAS

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Lie algebras play an increasingly important role in several areas of modern control theory and filtering and estimation theory, and also, in fact, in many (other) areas of (stochastic) mechanics. This is especially true for the approach to (nonlinear) filtering theory via the Duncan-Mortensen-Zakai equation (also called the reference probability approach). The material in this tutorial corresponds to the second lecture of my series of lectures on filtering. It forms a separate coherent unit and contains all the definitions concepts and results specific to Lie algebra theory that are needed (so far) in this approach. This tutorial is a revised and expanded version of an earlier one with the same title which appeared in M. Hazewinkel, J.C. Willems (eds), Stochastic systems: the mathematics of filtering and identification, Reidel 1981, 95-108.

1. Definition of Lie Algebras. Examples

Let k be a field and V a vectorspace over k. (For the purpose of this volume it suffices to take $k=\mathbf{R}$ or (rarely) $k=\mathbf{C}$; the vectorspace V over k need not be finite dimensional.) A Lie algebra structure on V is then a bilinear map (called bracket multiplication)

$$[\ ,\ ]: V\times V \to V \tag{1.1}$$

such that the two following conditions hold

$$[u,u]=0 \text{ for all } u\in V \tag{1.2}$$

$$[u,[v,w]]+[v,[w,u]]+[w,[u,v]]=0 \text{ for all } u,v,w,\in V. \tag{1.3}$$

The last identity is called the Jacobi identity. Of course the bilinearity of (1.1) means that $[au+bv,w]=a[u,w]+b[v,w]$, $[u,bv+cw]=b[u,v]+c[u,w]$. From (1.2) it follows that

$$[u,v]=-[v,u] \tag{1.4}$$

by considering $[u+v,u+v]=0$ and using bilinearity.

1.5. Example. The Lie algebra associated to an associative algebra
Let A be an associative algebra over k. Now define a new multiplication (bracket) on A by the formula

$$[v,w]=vw-wv,\ w,v\in A \tag{1.6}$$

Then A with this new multiplication is a Lie algebra. (Exercise: check the Jacobi identity (1.3).)

1.7. Remark
In a certain precise sense all Lie algebras arise in this way. That is for every Lie algebra L there is an associative algebra A containing L such that $[u,v]=uv-vu$. I.e. every Lie algebra arises as a subspace of an associative algebra A which happens to be closed under the operation $(u,v)\mapsto uv-vu$. Though this "universal enveloping algebra" construction is quite important it will play no role in the following and the remark is just intended to make Lie algebras easier to understand for the reader.

1.8. Example
Let $M_n(k)$ be the associative algebra of all $n \times n$ matrices with coefficients in k. The associated Lie algebra is written $gl_n(k)$; i.e. $gl_n(k)$ is the n^2-dimensional vectorspace of all $n \times n$ matrices with the bracket multiplication $[A,B] = AB - BA$.

1.9. Example
Let $sl_n(k)$ denote the subspace of all $n \times n$ matrices of trace zero. Because $Tr(AB - BA) = 0$ for all $n \times n$ matrices A,B, we see that $[A,B] \in sl_n(k)$ if $A,B \in sl_n(k)$ giving us an $(n^2 - 1)$-dimensional sub-Lie-algebra of $gl_n(k)$.

1.10. Example. The Lie algebra of first order differential operators with C^∞-coefficients.
Let V_n be the space of all differential operators (on the space $F(\mathbb{R}^n)$ of C^∞-functions (i.e. arbitrarily often differentiable functions in $x_1,...,x_n$)) of the form

$$X = \sum_{i=1}^{n} f_i(x_1,...,x_n)\frac{\partial}{\partial x_i} \tag{1.11}$$

where the $f_i, i = 1, \ldots, n$ are C^∞-functions. Thus $X: F(\mathbb{R}^n) \to F(\mathbb{R}^n)$ is the operator $X(\phi) = \sum_{i=1}^{n} f_i \frac{\partial \phi}{\partial x_i}$.
Now define a bracket operation on V_n by the formula

$$[X,Y] = \sum_{i,j} (f_i \frac{\partial g_j}{\partial x_i}\frac{\partial}{\partial x_j} - g_j \frac{\partial f_i}{\partial x_j}\frac{\partial}{\partial x_i}) \tag{1.12}$$

if $X = \sum f_i \frac{\partial}{\partial x_i}, Y = \sum g_j \frac{\partial}{\partial x_j}$. This makes V_n a Lie algebra. Check that $[X,Y](\phi) = X(Y(\phi)) - Y(X(\phi))$ for all $\phi \in F(\mathbb{R}^n)$.

More generally, cf the tutorial in differentiable manifolds and calculus in manifolds on this volume, one has the infinite dimensional Lie algebra of vectorfields $V(M)$ on any differentiable manifold M.

1.13. Example. Derivations
Let A be any algebra (i.e. A is a vectorspace together with any bilinear map (multiplication) $A \times A \to A$; in particular A need not be associative). A *derivation* on A is a linear map $D: A \to A$ such that

$$D(uv) = (Du)v + u(Dv) \tag{1.14}$$

For example let $A = \mathbb{R}[x]$ and D the operator $\frac{d}{dx}$. Then D is a derivation. The operators (1.11) of the example above are the derivations on $F(\mathbb{R}^n)$. Also in the case of the Lie algebra of smooth vectorfields $V(M)$ on a differentiable manifolds M, one has that $V(M)$ is the Lie algebra of derivations on the algebra $F(M)$ of smooth, i.e. C^∞, functions on M.

Let $Der(A)$ be the vectorspace of all derivations. Define $[D_1,D_2] = D_1D_2 - D_2D_1$. Then $[D_1,D_2]$ is again a derivation and this bracket multiplication makes $Der(A)$ a Lie algebra over k.

1.15. Example. The Weyl algebra W_1
Let W_1 be the vectorspace of all (any order) differential operators in one variable with polynomial coefficients. I.e. W_1 is the vectorspace with basis $x^i \frac{d^j}{dx^j}$, $i,j \in \mathbb{N} \cup \{0\}$. ($x^i$ is considered as the operator $f(x) \mapsto x^i f(x)$.) Consider W_1 as a space of operators acting, say, on $k[x]$. Composition of operators makes W_1 an associative algebra and hence gives W_1 also the structure of a Lie algebra; cf. example 1.5 above. For example one has

$$[x\frac{d^2}{dx^2}, x^2\frac{d}{dx}] = 5x^2\frac{d^2}{dx^2} + 2x\frac{d}{dx}, \quad [x\frac{d}{dx}, x^i\frac{d^j}{dx^j}] = (i-j)x^i\frac{d^j}{dx^j}$$

1.16. Example. The oscillator algebra
Consider the four dimensional subspace of W_1 spanned by the four operators $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2, x, \frac{d}{dx}, 1$.
One easily checks that (under the bracket multiplication of W_1)

$$
\begin{aligned}
&[\tfrac{1}{2}\tfrac{d^2}{dx^2} - \tfrac{1}{2}x^2, x] = \tfrac{d}{dx}, \quad [\tfrac{1}{2}\tfrac{d^2}{dx^2} - \tfrac{1}{2}x^2, \tfrac{d}{dx}] = x, \\
&[\tfrac{d}{dx}, x] = 1, \qquad\qquad (1.17) \\
&[\tfrac{1}{2}\tfrac{d^2}{dx^2} - \tfrac{1}{2}x^2, 1] = [x, 1] = [\tfrac{d}{dx}, 1] = 0
\end{aligned}
$$

Thus this four dimensional subspace is a sub-Lie-algebra of W_1. It is called the *oscillator Lie algebra* (being intimately associated to the harmonic oscillator).

2. Homomorphisms, isomorphisms, subalgebras and ideals

2.1. Sub-Lie-algebras
Let L be a Liealgebra over k and V a subvectorspace of L. If $[u,v] \in V$ for all $u,v \in L$, then V is a sub-Lie-algebra of L. We have already seen a number of examples of this, e.g. the oscillator algebra of example 1.16 as a sub-Lie-algebra of the Weyl algebra W_1 and the Lie-algebra $sl_n(k)$ as a sub-Lie-algebra of $gl_n(k)$. Some more examples follow.

2.2. The Lie-algebra $so_n(k)$.
Let $so_n(k)$ be the subspace of $gl_n(k)$ consisting of all matrices A such that $A + A^T = 0$ (where the upper ^{T} denotes transposes). Then if $A,B \in so_n(k)$ $[A,B] + [A,B]^T = AB - BA + (AB - BA)^T = A(B + B^T) - B(A + A^T) + (B^T + B)A^T - (A^T + A)B^T = 0$, so that $[A,B] \in so_n(k)$. Thus $so_n(k)$ is a sub-Lie-algebra of $gl_n(k)$.

2.3. The Lie-algebra $t_n(k)$.
Let $t_n(k)$ be the subspace of $gl_n(k)$ consisting of all upper triangular matrices. Because product and sum of upper triangular matrices are again upper triangular $t_n(k)$ is a sub-Lie-algebra of $gl_n(k)$.

2.4. The Lie-algebra $sp_n(k)$.
Let Q be the $2n \times 2n$ matrix $Q = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}$. Now let $sp_n(k)$ be the subspace of all $2n \times 2n$ matrices A such that $AQ + QA^T = 0$. Then as above in example 2.2 one sees that $A,B \in sp_n(k) \Rightarrow [A,B] \in sp_n(k)$ so that $sp_n(k)$ is a sub -Lie-algebra of $gl_{2n}(k)$.

2.5. Ideals
Let L be a Lie-algebra over k. A subvectorspace $I \subset L$ with the property that for all $u \in I$ and all $v \in L$ we have $[u,v] \in I$ is called an ideal of L. An example is $sl_n(k) \subset gl_n(k)$, cf example 1.8 above. Another example follows.

2.6. Example. The Heisenberg Lie-algebra
Consider the 3-dimensional subspace of W_1 spanned by the operators $x, \frac{d}{dx}, 1$. The formulas (1.17) show that this subspace is an ideal in the oscillator algebra.

2.7. Example. The centre of a Lie algebra
Let L be a Lie algebra. The centre of L defined as the subset $Z(L) = \{z \in L : [u,z] = 0 \text{ for all } u \in L\}$. Then $Z(L)$ is a subvector space of L and in fact an ideal of L. As an example it is easy to check that the centre of $gl_n(k)$ consists of scalar multiples of the unit matrix I_n.

2.8. Homomorphisms and isomorphisms.
Let L_1 and L_2 be a two Lie algebras over k. A morphism $\alpha: L_1 \to L_2$ of vectorpaces (i.e. a k-linear map) is a *homomorphism of Lie algebras* if $\alpha[u,v] = [\alpha(u),\alpha(v)]$ for all $u,v \in L_1$. The homomorphism α is called an isomorphism if it is also an isomorphism of vectorspaces.

2.9. Example
Consider the following three first-order differential operators in two variables x,P

$$a = (1-P^2)\frac{\partial}{\partial P} - Px\frac{\partial}{\partial x},\quad b = P\frac{\partial}{\partial x},\quad c = \frac{\partial}{\partial x}$$

Then one easily calculates (cf. (1.9)) $[a,b] = c$, $[a,c] = b$, $[b,c] = 0$. Now define α from the oscillator algebra of example 1.16 to this 3-dimensional Lie algebra as the linear map $\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{2}x^2 \mapsto a$, $x \mapsto b$, $\frac{d}{dx} \mapsto c$, $1 \mapsto 0$. Then the formulas above and (1.17) show that α is a homomorphism of Lie algebras.

2.10. Kernel of a homomorphism
Let $\alpha: L_1 \to L_2$ be a homomorphism of Lie algebras. Let $\mathrm{Ker}(\alpha) = \{u \in L_1 : \alpha(u) = 0\}$. Then $\ker(\alpha)$ is ideal in L_1.

2.11 Quotient Lie algebras
Let L be a algebra and I an ideal in L. Consider the quotient vector space L/I and the quotient morphisms of vector spaces $L \xrightarrow{\alpha} L/I$. For all $\bar{u},\bar{v} \in L/I$ choose $u,v \in L$ such that $\alpha(u) = \bar{u}$, $\alpha(v) = \bar{v}$. Now define $[\bar{u},\bar{v}] = \alpha[u,v]$. Check that this does not depend on the choice of u,v.

This then defines a Lie-algebra structure on L/I and $\alpha: L \to L/I$ becomes a homomorphism of Lie-algebras.

2.12. Image of a homomorphism
Let $\alpha: L_1 \to L_2$ be a homomorphism of Lie algebras. Let $\mathrm{Im}(\alpha) = \alpha(L_1) = \{u \in L_2 : \exists v \in L_1, \alpha(v) = u\}$. Then Im (α) is a sub-Lie-algebra of L_2 and α induces an isomorphism $L_1/Ker(\alpha) \cong \mathrm{Im}(\alpha)$.

2.13. Exercise
Consider the 3-dimensional vector space of all real upper triangular 3×3 matrices with zero's on diagonal. Show that this a sub-Lie-algebra of $gl_3(\mathbf{R})$, and show that it is isomorphic to the 3-dimensional Heisenberg-Lie-algebra of example 2.6 but that is not isomorphic to the 3-dimensional Lie-algebra $sl_2(\mathbf{R})$ of example 1.8.

2.14. Exercise
Show that the four operators x^2, $\frac{d^2}{dx^2}$, $x\frac{d}{dx}$, 1 span a 4-dimensional subalgebra of W_1, and show that this 4-dimensional Lie algebra contains a three dimensional Lie algebra which is isomorphic to $sl_2(\mathbf{R})$.

2.15. Exercise

Show that the six operators x^2, $\frac{d^2}{dx^2}$, x, $\frac{d}{dx}$, $x\frac{d}{dx}$, 1 span a six dimensional sub-Lie-algebra of W_1. Shows that $x, \frac{d}{dx}, 1$ span a 3-dimensional ideal in this Lie-algebra and shows that the corresponding quotient algebra is $sl_2(\mathbb{R})$.

3. LIE ALGEBRAS OF VECTORFIELDS

Let M be a C^∞-manifold (cf. the tutorial on manifolds and calculus on manifolds in this volume). Intuitively a vectorfield on M specifies a tangent vector $t(m)$ at every point $m \in M$. Then given a C^∞-function f on M we can for each $m \in M$ take the derivation of f at m in the direction $t(m)$, giving us a new function g on M. This can be made precise in varying ways; e.g. as follows.

3.1. The Lie algebra of vectorfields on a manifold M

Let M be a C^∞-manifold, and let $F(M)$ be the $\mathbb{R}$-algebra (pointwise addition and multiplication) of all smooth ($=C^\infty$) functions $f: M \to \mathbb{R}$. *By definition* a C^∞*-vectorfield* on M is a derivation $X: F(M) \to F(M)$. The Lie algebra of derivations of $F(M)$, cf. example 1.13, i.e. the Lie-algebra of smooth vectorfields on M, is denoted $V(M)$.

3.2. Derivations and vectorfields

Now let $M = \mathbb{R}^n$ so that $F(M)$ is simply the $\mathbb{R}$-algebra of C^∞-functions in $x_1, \ldots, x_n$. Then it is not difficult to show that every derivation $X: F(\mathbb{R}^n) \to F(\mathbb{R}^n)$ is necessarily of the form

$$X = \sum_{i=1}^{n} g_i \frac{\partial}{\partial x_i} \tag{3.3}$$

with $g_i \in F(\mathbb{R}^n)$. For a proof cf. [4, Ch.I, § 2]. The corresponding vectorfield on $\mathbb{R}^n$ now assigns to $x \in \mathbb{R}^n$ the tangent vector $(g_1(x), \ldots, g_n(x))^T$.

On an arbitrary manifold we have representations (3.3) locally around every point and these expressions turn out to be compatible in precisely the way needed to define a vectorfield as described in the tutorial on differentiable manifolds and calculus on manifolds in this volume.

3.4. Homomorphisms of Lie algebras of vectorfields

Let M and N be C^∞-manifolds and let $\alpha: L \to V(N)$ be a homomorphism of Lie algebras where L is a sub-Lie-algebra of $V(M)$. Let $\phi: M \to N$ be a smooth map. Then α and ϕ are said to be *compatible* if

$$\phi^*(\alpha(X)f) = X(\phi^*(f)) \quad \text{for all } f \in F(N) \tag{3.5}$$

where ϕ^* is the homomorphism of algebras $F(N) \to F(M), f \mapsto \phi^*(f) = f \circ \phi$.

In terms of the Jacobian of ϕ (cf. [3]), this means that

$$J(\phi)(X_m) = \alpha(X)_{\phi(m)} \tag{3.6}$$

where X_m is the tangent vector at m of the vectorfield X.

If $\phi: M \to N$ is an isomorphism of C^∞-manifolds there is always precisely one homomorphism of Lie-algebras $\alpha: V(M) \to V(N)$ compatible with ϕ (which is then an isomorphism). It is defined (via formula (3.5)) by

$$\alpha(X)(f) = (\phi^*)^{-1} X(\phi^* f), \quad f \in F(N). \tag{3.7}$$

3.8. Isotropy subalgebras
Let L be a sub-Lie-algebra of $V(M)$ and let $m \in M$. The isotropy subalgebra L_m of L at m consists of all vectorfields in L whose tangent vector vector in m is zero, or, equivalently

$$L_m = \{X \in L : Xf(m) = 0 \text{ all } f \in F(M)\} \tag{3.9}$$

Now suppose that $\alpha : L \to V(N)$ and $\phi : M \to N$ are compatible in the sense of 3.4 above. Then it follows easily from (3.5) that

$$\alpha(L_m) \subset V(N)_{\phi(m)} \tag{3.10}$$

i.e. α takes isotropy subalgebras into isotropy subalgebras. Inversely if we restrict our attention to analytic vectorfields then condition (3.10) on α at m implies that locally there exists a ϕ which is compatible with α [7].

4. Simple, Nilpotent, and solvable algebras

4.1. Nilpotent Lie algebras.
Let L be a Lie-algebra over k. The *descending central series* of L is defined inductively by

$$C^1L = L, \quad C^{i+1}L = [L, C^iL], \quad i \geq 1 \tag{4.2}$$

It is easy to check that the C^iL are ideals. The Lie algebra L is called *nilpotent* if $C^nL = \{0\}$ for n big enough.

For each $x \in L$ we have the endomorphism $adx : L \to L$ defined by $y \mapsto [x,y]$. It is now a theorem that if L is finite dimensional then L is nilpotent iff the endomorphisms adx are nilpotent for all $x \in L$. Whence the terminology.

4.3. Solvable Lie algebras.
The *derived series of Lie algebras* of a Lie algebra of a Lie algebra L is defined inductively by

$$D^1L = L, \quad D^{i+1}L = [D^iL, D^iL], \quad i \geq 1 \tag{4.4}$$

It is again easy to check that the D^iL are ideals. The Lie algebra L is called solvable if $D^nL = \{0\}$ for n large enough.

4.5. Examples
The Heisenberg Lie algebra of example 2.6 is nilpotent. The Oscillator algebra of example 1.16 is solvable but not nilpotent. The sub-Lie-algebra of W_1 with vector-space basis $x^2, \frac{d^2}{dx^2}, x, \frac{d}{dx}, 1, x\frac{d}{dx}$ is neither nilpotent, nor solvable. The Lie-algebra $t_n(k)$ of example 2.3 is solvable and in a way is typical of finite dimensional solvable Lie algebras in the sense that if k is algebraically closed (e.g. $K = \mathbf{C}$), then every finite dimensional solvable Lie algebra over k is isomorphic to a sub-Lie-algebra of some $t_n(k)$.

4.6. Exercise
Show that sub-Lie-algebras and quotient-Lie-algebras of solvable Lie algebras (resp. nilpotent Lie algebras) are solvable (resp. nilpotent).

4.7. Abelian Lie-algebras
A Lie algebra L is called *abelian* if $[L,L] = \{0\}$, i.e. if every bracket product is zero.

4.8. Simple Lie-algebras
A Lie algebra L is called *simple* if it is not abelian and if it has no other ideals than 0 and L. (Given the second condition the first one rules out the zero- and one-dimensional Lie algebras.) These simple-Lie-algebras and the abelian ones are in a very precise sense the basic building blocks of all Lie algebras.

The finite dimensional simple Lie algebras over $\mathbb{C}$ have been classified. They are the Lie algebras $sl_n(\mathbb{C}), sp_n(\mathbb{C}), so_n(\mathbb{C})$ of examples 1.8, 2.4 and 2.2 above and five additional exceptional Lie algebras. For infinite dimensional Lie algebras things are more complicated. The socalled filtered, primitive, transitive simple Lie algebras have also been classified (cf. e.g. [2]). One of these is the Lie-algebra $\tilde{V}_n$ of all formal vector fields $\sum f_i(x_i, \ldots, x_n)\frac{\partial}{\partial x_i}$, where the $f_i(x)$ are (possibly non converging) formal power series in $x_1, \ldots, x_n$. This class of infinite dimensional simple Lie algebras by no means exhausts all possibilities. E.g. the quotient-Lie algebras $W_n/\mathbb{R}\cdot 1$ are simple and non-isomorphic to any of those just mentioned.

4.9. Exercise
Let $V_{alg}(\mathbb{R}^n)$ be the Lie algebra of all differential operators (vector fields) of the form $\sum f_i(x_1, \ldots, x_n)\frac{\partial}{\partial x_1}$ with $f_i(x_1, \ldots, x_n)$ polynomial. Prove that $V_{alg}(\mathbb{R}^n)$ is simple.

5. Representations
Let L be a Lie algebra over k and M a vectorspace over k. A representation of L in M is a homomorphism of Lie algebras.

$$\rho: L \to End_k(M) \tag{5.1}$$

where $End_k(M)$ is the vectorspace of all k-linear maps $M \to M$ which is of course given the Lie algebra structure $[A,B]=AB-BA$. Equivalently a representation of L in M consists of a k-bilinear map

$$\sigma: L \times M \to M \tag{5.2}$$

such that, writing xm for $\sigma(x,m)$, we have $[x,y]m = x(ym) - y(xm)$ for all $x,y \in L$, $m \in M$. The relation between the two definitions is of course $\sigma(x,m) = \rho(x)(m)$.

Instead of speaking of a representation of L in M we also speak (equivalently) of the L-module M.

5.3. Examples
The Lie algebra $gl_n(k)$ of all $n \times n$ matrices naturally acts on k^n by $(A,v) \mapsto Av \in k^n$ and this defines a representation $gl_n(k) \times k^n \to k^n$. The Lie algebra $V(M)$ of vectorfields on a manifold M acts (by its definition) on $F(M)$ and this is a representation of $V(M)$. A quite important theorem concerning the existence of representations is

5.4. Ado's theorem
Cf. e.g. [1, § 7]. If k is a field of characteristic zero, e.g. $k = \mathbb{R}$ or $\mathbb{C}$ and L is finite dimensional then there is a faithful representation $\rho: L \to End(k^n)$ for some n. (Here faithful means that ρ is injective.)

Thus every finite dimensional Lie algebra L over k (of characteristic zero) can be viewed as a subalgebra of some $gl_n(k)$, and this subalgebra can then be viewed as a more concrete matrix "representation" of the "abstract" Lie algebra L.

5.5. Realizing Lie-algebras in $V(M)$
A question of some importance for filtering theory is when a Lie algebra L can be realized as a sub-Lie-algebra of $V(M)$, i.e. when L can be represented in $F(M)$ by means of derivations. For finite dimensional Lie algebras Ado's theorem gives the answer because $(a_{ij}) \mapsto \sum a_{ij} x_i \frac{\partial}{\partial x_j}$ defines an injective homomorphism of Lie-algebras $gl_n(\mathbb{R}) \to V(\mathbb{R}^n)$ (Exercise: check this).

6. LIE ALGEBRAS AND LIE GROUPS

6.1. Lie groups
A (finite dimensional) Lie group is a finite dimensional smooth manifold G together with smooth maps $G \times G \to G$, $(x,y) \mapsto xy$, $G \to G$, $x \mapsto x^{-1}$ and a distinguished element $e \in G$ which make G a group. An example is the open subset of $\mathbb{R}^{n^2}$ consisting of all invertible $n \times n$ matrices with the usual matrix multiplication.

6.2. Left invariant vectorfields and the Lie algebra of a Lie group
Let G be a Lie group. Let for all $g \in G$, $L_g: G \to G$ be the smooth map $x \to gx$. A vectorfield $X \in V(G)$ is called left invariant if $X(L_g^* f) = L_g^*(Xf)$ for all functions f on G, where, of course, the left translate $L_g^* \gamma$ of a function γ is defined by $(L_g^* \gamma)(x) = \gamma(gx)$. Or, equivalently, if $J(L_g)X_x = X_{gx}$ for all $x \in G$, cf. section 3.4 above. Especially from the last condition it is easy to see that $X \mapsto X_e$ defines an isomorphism between the vectorspace of left invariant vectorfields on G and the tangent space of G at e. Now the bracket product of two left invariant vectorfields is easily seen to be left invariant again so the tangent space of G at e (which is $\mathbb{R}^n$ if G is n-dimensional) inherits a Lie algebra structure. This is the Lie algebra Lie(G) of the Lie group G. A main reason for the importance of Lie algebras in many parts of mathematics and its applications is that this construction is reversible to a great extent making it possible to study Lie groups by means of their Lie algebras.

6.3. Exercise
Show that the Lie algebra of the Lie group $GL_n(\mathbb{R})$ of invertible real $n \times n$ matrices is the Lie algebra $gl_n(\mathbb{R})$.

7. THE ADJOINT REPRESENTATION
Let L be a Lie algebra. Then there is a natural representation of L into the vectorspace L given by

$$ad: L \to End(L), \quad ad(x)(y) = [x,y] \tag{7.1}$$

The Jacobi identity is precisely what is needed to show that $ad([x,y]) = ad(x)ad(y) - ad(y)ad(x)$. This representation is called the adjoint representation. It is the infinitesimal part of a representation denoted Ad of the group G of L in L.

In the case G is a connected subgroup of $GL_n(k)$, so that $L = \text{Lie}(G)$ is a subalgebra of $gl_n(k)$ this representation can be written as

$$Ad(g)(x) = gxg^{-1}, \quad x \in L, g \in G \tag{7.2}$$

(One needs to prove of course that gxg^{-1} is again in L.) In the more general and more abstract setting of 6.2 above, this goes as follows. Let G be a Lie group and L its Lie algebra, i.e. the Lie algebra of left invariant smooth vectorfields on G. Again let $L_g: G \to G$ be defined by $L_g(x) = gx$ and define $i_h: G \to G$ by $x \mapsto hxh^{-1}$. Now observe that

$$L_g i_h = i_h L_{h^{-1}gh}$$

It follows immediately that if the vectorfield $i_h^* X$ is defined by

$$(i_h^* X)(f) = X(i_h^* f)$$

that then i_h^* is left invariant if X is left-invariant.

In case $G=GL_n(\mathbb{R})\subset\mathbb{R}^{n\times n}$, we can identity the tangent space T_gG at $g\in G$ with the space of $n\times n$ matrices $\mathbb{R}^{n\times n}$. The left invariant vectorfields then are of the form

$$g\mapsto gA,\quad g\in G$$

for some fixed $A\in\mathbb{R}$. If X is a vectorfield then i_h^*X is the vectorfield defined by

$$(i_h^*X)_x=J(i_h)(i_h^{-1}(x))(X_{i_h^{-1}(x)})$$

Now $i_h(x)=hxh^{-1}$ which is linear in x. So $J(i_h)(y)(v)=hvh^{-1}$ for all $y\in G$, $v\in\mathbb{R}^{n\times n}=T_yG$. So if X is the left invariant vectorfield $g\mapsto gA$, then i_h^* is the vectorfield

$$(i_h^*X)_g=J(i_h)(i_h^{-1}(g))(X_{i_h^{-1}(g)}) = J(i_h)(i_h^{-1}(g))(h^{-1}ghA)$$
$$= h(h^{-1}ghA)h^{-1}=ghAh^{-1}$$

which is indeed again of the same form, hence left invariant, and we see that the induced action on $T_eG=\mathbb{R}^{n\times n}$ is indeed $A\mapsto hAh^{-1}$. This also serves to prove that if $g\in G\subset GL_n(\mathbb{R})$, $x\in L\subset gl_n(\mathbb{R}), L=$ Lie(G) then indeed $gxg^{-1}\in L$.

For $G=GL_n(k)$, $L=gl_n(k)$ there is a local diffeomorphism of a neighborhood of 0 in $gl_n(k)$ to a neighborhood of $e=I_n\in GL_n(k)$ given by $A\mapsto \exp(A)=I+\frac{A}{1!}+\frac{A^2}{2!}+\cdots$ and more generally exp takes a neighborhood in the subalgebra L of G into one around $e\in G$.

The particular case of (7.2) where g is of the form $e^A, A\in L$, is of importance. One has *'the adjoint representation formula'*

$$e^ABe^{-A}=B+\frac{[A,B]}{1!}+\frac{[A,[A,B]]}{2!}+\cdots=\sum_{n=0}^{\infty}(n!)^{-1}ad(A)^n(B) \tag{7.3}$$

(From which it is clear that indeed $e^ABe^{-A}\in$ (a completion of) the Lie algebra generated by A and B). This formula is easily proved by induction. One needs $ad(A)^n(B)=\sum_{i+j=n}(-1)^j\frac{1}{i!}\cdot\frac{1}{j!}A^iBA^j$. Occasionally in the literature this formula occurs under the name Campbell-Baker-Hausdorff formula (or Campbell-Hausdorff formula). This name, however, belongs more properly to the far deeper result that if $A,B\in L$ then e^Ae^B is the exponential of some element in (a completion of) the sub-Lie-algebra of L spanned by A and B. This element can be expressed as an infinite sum (the $C-B-H$ formula)

$$A+B+\frac{1}{2}[A,B]+\frac{1}{12}[[A,B],B]-\frac{1}{12}[[A,B],A]+\cdots$$

(This result can be extended to the most general case: that of a free Lie algebra, either in terms of formal series identities, or by means of suitable completions.)

8. Postscript

The above is a very rudimentary introduction to Lie algebras. Especially the topic "Lie algebras and Lie groups" also called "Lie theory" has been given very little space, in spite of the fact that it is likely to become of some importance in filtering (integration of a representation of a Lie algebra to a representation of a Lie (semi)group). The books [1, 4, 5, 6, 8, 9] are all recommended for further material. My current personal favourite is [9] with [4] as (a far more difficult) close runner-up; [6] is a classic and in its present incarnation very good value indeed.

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